Math 261B Tues. 9/8/2020

$A = \mathcal{O}(G)$ is an A-comodule via $\underset{\substack{\| \\ V}}{\Delta : A} \to \underset{\substack{\| \\ V}}{A \otimes A}$

$A \curvearrowleft G$ $\qquad G \curvearrowright G \curvearrowleft G$

make left by $g \cdot h = hg^{-1}$

$f \cdot g = f \circ r_{g^{-1}}$ $\qquad g \mapsto r_{g^{-1}}$

$f \in A$, $g \in G$

$G \curvearrowright A \quad g \cdot f = f \circ r_g$ $\qquad$ (all comodules are locally finite)

$(g \cdot f)(h) = f(hg)$

Ex. $\mathbb{G}_a \quad A = k[z] \qquad \langle 1, z, \ldots, z^m \rangle$ is $\mathbb{G}_a$-invariant

$x \cdot f(z) = f(z + x)$, $x \in \mathbb{G}_a$

$x \cdot z^k = z^k + (\text{lower terms})$

$$x \longmapsto \begin{pmatrix} 1 & * & * & & \\ & 1 & * & & * \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$$

upper uni-triangular

Only irreducible rep. seen in A for $\mathbb{G}_a$ is $k\cdot 1$ $\leftarrow$ trivial rep.

no proper non-zero invariant subspace

<u>Ex.</u> $\mathbb{G}_m$ $\quad A = k[t^{\pm 1}]$ $\quad \langle t^m \rangle$ is invariant and irreducible

$$a \cdot t^m = (ta)^m = a^m t^m \qquad a \to (a^m)$$

A is a $\oplus$ of irreducible submodules.

Say $G \curvearrowright V \cong k^n$ $\qquad G \to GL(V)$

$$g \mapsto M(g) = \Big(M_{ij}(g) \Big)$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in A$

$G \curvearrowright A$ by $g \cdot f = f \circ r_g$

$$\Rightarrow (g \cdot M_{ij})(h) = M_{ij}(hg) = \sum_k M_{ik}(h) M_{kj}(g)$$

$$g \cdot M_{ij} = \sum_k M_{kj}(g) \; M_{ik}$$

Basis vectors $v_1, \dots, v_n$ of V

$$g v_j \longleftrightarrow M(g) e_j = \sum_k M_{kj}(g) e_k \longleftrightarrow \sum_k M_{kj}(g) v_k$$

Matches g action on M_{ij} for fixed i.

Get $V \underset{\cong}{\xrightarrow{\varphi}} W$ = G-invariant subspace of A spanned by $M_{i,-}$

G-homomorphism

$\varphi \neq 0$ because $M_{ii}(e) = 1$

If V is <u>irreducible</u> then $\ker(\varphi) = 0$, i.e. φ is injective, hence $\cong$

($\dim W \leq \dim V$)

This shows

<u>Prop.</u> Every fin. dim'l irr. alg. representation of G // linear alg gp. occurs as a submodule of $A = \mathcal{O}(G)$.

<u>Cor.</u> TFAE:

(i) Every G invariant subspace of A contains $k \cdot 1$

(ii) The only f. dim. irr. rep. of G is the trivial $G \curvearrowright K$

(iii) Every fin. dim. rep. V of G is upper uni-Δar in some basis of V

(iv) $G \curvearrowright A$ by upper uni-Δar matrices in some basis of A.

(v) $\exists$ $G \hookrightarrow U \leq GL_n$

upper uni-Δar matrices

$V \quad V' = V/kv$

$\cup \quad\quad \cup$

$v \quad\quad v' \ldots$

$gv = v$ for all $g \in G$ $\quad gv' = v' + a(g)v$

(v) $\Rightarrow$ (i) Enough to do case $G = U$ $A = k[x_{ij} \mid i<j]$

$g\cdot f = f\circ r_g$
$$\begin{pmatrix} 1 & x_{12} & x_{13} & \\ & \ddots & & \ddots \\ 0 & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \ddots & g_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_{12}+g_{12} & x_{13}+x_{12}g_{23} & \dots \\ & \ddots & & \\ & & & 1 \end{pmatrix}$$

Can order monomials in x_{ij}'s so $G \curvearrowright A$ is upper uni-triangular.

Unipotent group

Ex. U, subgroups of U, $\mathbb{G}_a \cong U$ in GL_2 $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

If $H \subset G$ closed normal subgroup, H unipotent, G/H unipotent $\Rightarrow$ G unipotent.

$\Rightarrow$ Every G has a unique largest unipotent closed normal group R_u

$H, K \subseteq G$ $H \lhd H\cdot K$ H, $HK/H = K/\dots$ unipotent $\Rightarrow$ $H\cdot K$ unipotent, $H\times K \xrightarrow{\mu} G$ closed, normal

R_u is the unipotent radical of G.

Remark In GL_n, have Jordan decomposition $X = X_s X_u$ ← commute

exists by Jordan form
(and unique)

$$\begin{pmatrix} \begin{pmatrix} c & & 0 \\ & \ddots & \\ & & c \end{pmatrix} & & \\ & \boxed{} & \\ & & \ddots \end{pmatrix} \qquad \begin{pmatrix} c & & \\ & \ddots & \\ & & c \\ & & & \ddots \end{pmatrix}$$

X_s: semisimple (= diagonalizable)

X_u: unipotent (all evals $= 1$)

independent of $G \hookrightarrow GL_n$

Decomposition is preserved by any closed $G \subseteq GL_n$ $\qquad g = g_s g_u$

$X \in G \Rightarrow X_s, X_u \in G$.

G is unipotent $\Leftrightarrow$ $g = g_u$ for all $g \in G$.

Contrast: $\mathbb{G}_m$ $\qquad A = k[t^{\pm 1}] = \bigoplus \langle t^m \rangle$

R = projection on $\langle t^0 \rangle = k \cdot 1$ $\qquad$ R is G-invariant projection

$A \to k \cdot 1$

$R \cdot 1 = 1$

$\ker R$ G-invariant

$R \in A^*$ $\Rightarrow$ R acts on any A comodule.

$$V \xrightarrow{\rho} V \otimes A \xrightarrow{\mathrm{id}_V \otimes R} V \otimes k = V$$

Lemma R acts on V as a projection on V^G: $\qquad Rv = v$ if $v \in V^G$

$Rv \in V^G$ for all v

$$V = V^G \oplus \ker(R)$$

$\ker(R)$ is G-invariant

$\left(\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times \quad R = \int_{U_1} \right.$ measure 1

$U_1 = \{ z \in \mathbb{C}^\times \mid |z| = 1 \}$

$z^m \mapsto 0 \quad m \neq 0$

$z^0 \mapsto 1$

$e^{im\theta}$

$\left. \text{Finite } G: \quad R = \frac{1}{|G|} \sum_{g \in G} g \right)$

Proof $\quad v \in V^G$

$\Updownarrow$

$\rho(v) = v \otimes 1_A$

$A^\vee$

$\downarrow$

$Rv = \sum v_i \cdot Rf_i$

$v \cdot R(1)$

$\overset{\shortparallel}{1}$

$= v$

$\rho(Rv) \underset{\text{wts}}{=} Rv \otimes 1_A$

$V \xrightarrow{\rho} V \otimes A$

$v \mapsto \sum v_i \otimes f_i \quad f_i \in A$

$gv \mapsto \sum v_i \otimes f_i(g)$

v_i come from a basis of v

$R \mid V^G$ is identity

$\rho \otimes id_A = id_V \otimes \Delta$

$V \xrightarrow{\rho} V \otimes A \longrightarrow V \otimes A \otimes A$

$\rho^{(2)}$

$v \longmapsto \sum v_i \otimes f_{i_1} \otimes f_{i_2}$

$\rho(Rv) = \sum v_i \otimes f_{i1} \cdot Rf_{i2} \overset{?}{=} \sum v_i \otimes Rf_i$

$f \underset{\Delta}{\mapsto} \sum f_{(1)} \otimes f_{(2)} \qquad \sum f_{(1)} Rf_{(2)} \overset{?}{=} Rf \qquad f^m$

$(id \otimes R) \circ \Delta \overset{?}{=} R$

$(\mathrm{id} \otimes R) \circ \Delta \overset{?}{=} \overline{R}$ $\quad t^m$: $\quad$ RHS $= 0$ if $m \neq 0$

$R(1) = 1$ $\quad$ LHS $\quad t^m \otimes t^m \xrightarrow[1 \otimes R]{} t^m \otimes 0 = 0$

i.e. R ($\in A^*$, acting on A) $\quad t^0$: $\quad (\mathrm{id} \otimes R)\, \Delta \cdot 1 = (\mathrm{id} \otimes R)(1 \otimes 1) = 1$

is projection on $A^G = k \cdot 1$. $\quad R(1) = 1$

$\exists$ Reynolds operator $\Rightarrow$ Complete reducibility, i.e. every fin dim'l V is a $\oplus$ of irreducibles

Given $W \subseteq V$ (G-submodule), want $V = W \oplus W'$ (G-submodule)

Or: find $\pi : V \to W$ (G invariant) s.t. $\pi|W = \mathrm{id}_W$ $\quad W' = \ker \pi$

$\pi : V \xrightarrow[\pi]{} W \xrightarrow[\pi]{\mathrm{id}} W$

$v = \underset{\substack{\cap \\ W}}{\pi v} + \underset{\substack{\cap \\ W'}}{(v - \pi v)}$ $\quad \pi^2 = \pi$

Pick $\tilde{\pi} \in \mathrm{Hom}_k(V, W)$ $\quad \pi(v - \pi v)$

s.t. $\tilde{\pi}|W = \mathrm{id}$ $\quad = \pi v - \pi^2 v = 0$

$\pi \,|\, W \cap W'$ is id, is 0 $\quad W \cap W' = 0$

G acts on V, W, $\mathrm{Hom}(V, W)$ $\quad V \otimes W \quad V^*$

$g \cdot \varphi = g \circ \varphi \circ g^{-1}$

V, W $\quad A$-modules

$\Rightarrow \Delta, S \ldots \rightsquigarrow V \otimes W$

$\tilde{\pi} \in Hom(V,W)$

$\pi = R\tilde{\pi} \in Hom(V,W)^G$

$\pi \in Hom_G(V,W)$

$g \cdot \varphi = \varphi$

$g \circ \varphi = \varphi \circ g \quad \forall g \in G$

φ is a G-homomorphism

$\tilde{\pi} \longmapsto \pi$

$$\begin{array}{ccccc} \tilde{\pi} \in Hom(V,W) & \xrightarrow{R} & H(V,W)^G & & \pi \\ \downarrow \quad \downarrow res_W & & \downarrow res_W & & \downarrow \\ id \in Hom(W,W) & \xrightarrow{R} & H(W,W)^G & & id \\ id & \longrightarrow & id & & \end{array}$$

commutes

$A \otimes A$

$\uparrow \Delta$

A

$g(v \otimes w) = gv \otimes gw$

$R \in A^*$

res_W is a comodule homomorphism