

Math 261B Tues. 9/8/2020

$A = \mathcal{O}(G)$ is an A -comodule via $\Delta: A \rightarrow A \otimes A$

$A \curvearrowright G$ $G \curvearrowright G \curvearrowright G$
 \uparrow make left by $g \cdot h = hg^{-1}$
 $g \mapsto r_{g^{-1}}$

$f \cdot g = f \circ r_{g^{-1}}$

$G \curvearrowright A$ $g \cdot f = f \circ r_g$ (all comodules are locally finite)
 $(g \cdot f)(h) = f(hg)$

Ex. G_a $A = k[z]$ $\langle 1, z, \dots, z^m \rangle$ is G_a -invariant

$$\begin{matrix} x \\ \uparrow \\ G_a \end{matrix} \cdot f(z) = f(z+x)$$

$$x \cdot z^k = z^k + (\text{lower terms})$$

upper uni-triangular

$$x \mapsto \begin{pmatrix} 1 & * & * & & \\ & 1 & * & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Only irreducible rep. seen in A for G_A is $k \cdot 1 \leftarrow$ trivial rep.
 no proper non-zero invariant subspace

Ex. $G_m \quad A = k[t^{\pm 1}] \quad (t^m)$ is invariant and irreducible
 $a \cdot t^m = (ta)^m = a^m t^m \quad a \rightarrow (a^m)$
 A is a \oplus of irreducible submodules.

Say $G \curvearrowright V \cong \underline{k^n}$

$G \rightarrow GL(V)$

$G \curvearrowright A$ by $g \cdot f = f \circ r_g$

$g \mapsto M(g) = \begin{pmatrix} M_{ij}(g) \\ \uparrow \\ A \end{pmatrix}$

$$\Rightarrow (g \cdot M_{ij})(h) = M_{ij}(hg) = \sum_k M_{ik}(h) M_{kj}(g)$$

$$g \cdot M_{ij} = \sum_k M_{kj}(g) M_{ik}$$

Basis vectors v_1, \dots, v_n of V

$$g v_j \leftrightarrow M(g) e_j = \sum_k M_{kj}(g) e_k \leftrightarrow \sum_k M_{kj}(g) v_k$$

Matches g action on M_{ij} for fixed i .

Get $V \xrightarrow{\varphi} W = G$ -invariant subspace of A spanned by $M_{i,j}$ -
 G -homomorphism

$\varphi \neq 0$ because $M_{i,i}(e) = 1$

If V is irreducible then $\ker(\varphi) = 0$, i.e. φ is injective, hence $\dim W = \dim V$
 ($\dim W \leq \dim V$)

This shows

Prop. Every fin. dim'd irr. alg. representation of G // linear alg. grp. occurs as a submodule of $A = \mathcal{O}(G)$.

Cor. TFAE: (i) Every G invariant subspace of A contains $k \cdot 1$

(ii) The only fin. dim. irr. rep. of G is the trivial G on k

(iii) Every fin. dim. rep. V of G is upper uni-tri in some basis of V

(iv) $G \curvearrowright A$ by upper uni-tri matrices in some basis of A .

(v) $\exists G \hookrightarrow U \subseteq GL_n$
 \uparrow
 upper uni-tri matrices

$V \quad V' = V/kv$
 $\cap \quad \cap$
 $v \quad v'$

$gv = v$
 for all $g \in G$ $gv' = v' + a(g)v$

(v) \Rightarrow (i) Enough to do case $G = U$ $A = k[x_{ij} \mid i < j]$

$$g \cdot f = f \circ r_g \quad \begin{pmatrix} 1 & x_{12} & x_{13} \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & g_{12} \\ & \ddots & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_{12} + g_{12} & x_{13} + x_{12}g_{13} \\ & \ddots & \\ & & 1 \end{pmatrix}$$

Can order monomials $\hookrightarrow x_{ij}$'s so $G \simeq A$ is upper unitriangular.

Unipotent group

EX. U , subgroups of U , $G_n \cong U$ in $GL_2 \begin{pmatrix} x & \\ 0 & 1 \end{pmatrix}$

If $H \subset G$ closed normal subgroup, H unipotent, G/H unipotent $\Rightarrow G$ unipotent.

\Rightarrow Every G has a unique largest unipotent closed normal group (R_u)

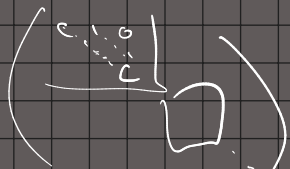
$H, K \in G$ $H \triangleleft H \cdot K$ $H, H \cdot K / H$ unipotent $\Rightarrow H/K$ unipotent,

R_u is the unipotent radical of G . $\begin{matrix} H & \times & K & \xrightarrow{H} & G \\ & & & & \text{closed, normal} \end{matrix}$

Remark

In GL_n , have Jordan decomposition $X = X_s X_u$

exists by Jordan form (and unique)



semisimple (= diagonalizable)

unipotent (all evals = 1)

independent of $G \hookrightarrow GL_n$

Decomposition is preserved by any closed $G \subseteq GL_n$

$$g = g_s g_u$$

$$X \in G \Rightarrow X_s, X_u \in G.$$

G is unipotent $\Leftrightarrow g = g_u$ for all $g \in G$.

Contrast: $G_m \quad A = k[t^{\pm 1}] = \bigoplus \langle t^m \rangle$

$R =$ projection on $\langle t^0 \rangle = k \cdot 1$

R is G -invariant projection

$R \in A^*$ $\Rightarrow R$ acts on any A module.

$$V \xrightarrow{p} V \otimes A \xrightarrow{id_V \otimes R} V \otimes k = V$$

$$\begin{aligned} A &\rightarrow k \cdot 1 \\ R \cdot 1 &= 1 \\ \text{ker } R & \text{ } G\text{-invariant} \end{aligned}$$

Lemma R acts on V as a projection on V^G :

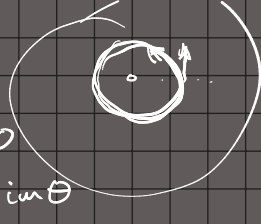
$$Rv = v \text{ if } v \in V^G$$

$$Rv \in V^G \text{ for all } v$$

$\text{ker}(R)$ is G -invariant

$$V = V^G \oplus \text{ker}(R)$$

$$G_m(\mathbb{C}) = \mathbb{C}^\times \quad R = \int_{\mu_1} \text{measure } 1 \quad \mu_1 = \{z \in \mathbb{C}^\times \mid |z|=1\}$$

$z^m \mapsto 0 \quad m \neq 0$
 $z^0 \mapsto 1$
 $e^{im\theta}$


$$\text{Finite } G: \quad R = \frac{1}{|G|} \sum_{g \in G} g$$

Proof $v \in V^G$

$$p(v) = v \otimes 1_A$$

$$Rv = \sum v_i \cdot Rf_i$$

$$V \xrightarrow{p} V \otimes A$$

$$v \mapsto \sum v_i \otimes f_i$$

$$gv \mapsto \sum v_i \otimes f_i(g)$$

v_i come from a basis of V

$$v \cdot R(1) = v$$

$$p(Rv) \stackrel{\text{WTS}}{=} Rv \otimes 1_A$$

$R \mid V^G$ is identity
 $p \circ \Delta = \text{id}_V \otimes \Delta$

$$V \xrightarrow{p} V \otimes A \xrightarrow{\Delta} V \otimes A \otimes A$$

$$v \mapsto \sum v_i \otimes f_i \otimes f_i$$

$$p(Rv) = \sum v_i \otimes f_{i1} \cdot Rf_{i2} \stackrel{?}{=} \sum v_i \otimes Rf_i$$

$$\uparrow \xrightarrow{\Delta} \sum f_{i1} \otimes f_{i2} \quad \sum f_{i1} Rf_{i2} \stackrel{?}{=} Rf \quad \uparrow$$

$$(\text{id} \otimes R) \circ \Delta = R$$

$$(id \otimes R) \circ \Delta = R$$

$R(1) = 1$

$$t^m : \text{RHS} = 0 \text{ if } m \neq 0$$

$$\text{LHS } t^m \otimes t^m \xrightarrow{id \otimes R} t^m \otimes 0 = 0$$

$$(id \otimes R) \Delta \cdot 1 = (id \otimes R) (1 \otimes 1) = 1$$

$R(1) = 1$

i.e. $R \in A^*$, acting on A t^0 is projection on $A^e = k \cdot 1$.

$\rightarrow \exists$ Reynolds operator \Rightarrow Complete reducibility, i.e. every fin dim'l V is a \oplus of irreducibles

G -submodule

Given $W \subseteq V$, want $V = W \oplus W'$ G -submodule

Or: find $\pi : V \rightarrow W$ s.t. $\pi|_W = id_W$ $W' = \ker \pi$

G invariant

$$\pi : V \rightarrow W \xrightarrow{id} W$$

$$v = \underbrace{\pi v}_W + \underbrace{(v - \pi v)}_{W'}$$

$$\pi^2 = \pi$$

$$\pi(v - \pi v) = \pi v - \pi^2 v = 0$$

$W \cap W' = 0$

Pick $\tilde{\pi} \in \text{Hom}_k(V, W)$ s.t. $\tilde{\pi}|_W = id$

G acts on V, W , $\text{Hom}(V, W)$

$$g \cdot \phi = g \circ \phi \circ g^{-1}$$

$$V \otimes W \quad V^*$$

$$\Rightarrow \Delta, S, \dots \rightsquigarrow V \otimes W$$

A -modules

$$\tilde{\pi} \in \text{Hom}(V, W)$$

$$\pi = \mathcal{R}\tilde{\pi} \in \text{Hom}(V, W)^G$$

$$\pi \in \text{Hom}_G(V, W)$$

 $\tilde{\pi}$

$$\tilde{\pi} \in \text{Hom}(V, W)$$

 \downarrow

$$\lambda \in \text{Hom}(W, W)$$

 id
 $\downarrow \text{res}_W$
 \mapsto
 \mathcal{R}
 \mathcal{R}
 \mapsto

$$g \cdot \phi = \phi$$

$$g \circ \phi = \phi \circ g \quad \forall g \in G$$

 ϕ is a G -homomorphism

$$H(V, W)^G$$

 $\downarrow \text{res}_W$

$$H(W, W)^G$$

 id
 π
 id
 \mathcal{R} commutes

 res_W is a

 G -homomorphism

$$\begin{array}{c} \mathcal{R} \\ A \otimes A \\ \downarrow \Delta \\ A \end{array}$$

$$g(v \otimes w) = gv \otimes gw$$

$$\mathcal{R} \in A^*$$